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# AN ALGORITHM FOR ACCURATELY ESTIMATING THE HARMONIC MAGNITUDES OF PERIODIC ARBITRARY SIGNALS USING ASYNCHRONOUS SAMPLING

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## Abstract

An algorithm that uses an integrating voltmeter for accurately estimating the harmonic magnitudes of periodic arbitrary signals is presented. The uncertainties associated with the magnitude estimates relative to the fundamental depend on signal stability, harmonic content, noise variance and are less than $1\times10^{-5}$ for signals with up to 64 harmonics.

## Introduction

An extension of Swerlein's algorithm[1][2] for accurately measuring the magnitudes of the harmonics of a low-frequency, low-distortion, voltage signal was already published [3]. It is shown in [4] that this extension allows one to obtain results with asynchronous sampling that numerically approach those obtained using synchronous sampling [5]. The contribution of this paper is to further extend the algorithm version based on discrete Fourier transforms described in [3] for accurately measuring the harmonic magnitudes of periodic nonsinusoidal signals.

## Model

A total of $n$ bursts of $N$ samples are taken. The internal level trigger of the digital voltmeter (DVM) is used to start each burst delayed by $kt_D$ ($k$ = 0, …, $n$–1) from a signal null-crossing. It is assumed that each burst can be modelled by

$$\mathbf{y}_k = \mathbf{W}_k \mathbf{x}\,, \tag{1}$$

where $\mathbf{y}_k = (y_{1k}, \ldots, y_{Nk})'$ is the data vector at the $k$-th burst, $\mathbf{W}_k$ is the known $N \times 2m$ matrix with ($i$, $j$)-th element $\cos 2\pi j f_0(t_i+kt_D)$ for $j$ = 1, …, $m$ and $\sin 2\pi(j-m)f_0(t_i+kt_D)$ for $j = m+1, \ldots, 2m$, $\mathbf{x}$ is the $2m$-vector of fitting parameters (uncorrected for the systematic effects), $jf_0$ is the $j$-th harmonic ($j$ = 1, …, $m$) of the known constant fundamental frequency $f_0$, and $m$ is the specified number of harmonics of the Fourier series.

The best estimate of $\mathbf{x}$ is [4]

$$\mathbf{x} = \left(\sum_{k=0}^{n-1} \mathbf{F}_k\right)^{-1} \sum_{k=0}^{n-1} \mathbf{W}_k' \mathbf{y}_k \tag{2}$$

where $\mathbf{F}_k = \mathbf{W}_k'\mathbf{W}_k$.

The error matrix $\Lambda_k = \mathbf{F}_k - (N/2)\mathbf{I}_{2m}$ for each value of $kt_D$, where $\mathbf{I}_{2m}$ is the identity matrix of order $2m$, can be nearly nullified as the algorithm tries to make $N \cdot t_{samp}$ (where $t_{samp}$ is the sampling period) equal to an integer number of periods, so that $\mathbf{F}_k$ becomes diagonal. The algorithm designs the experiment so that the matrix resulting from the first summation in (2) is diagonalized. If one chooses $t_D$ nearly equal to $1/nf_0$ and the average of $\mathbf{F}_k$ over all $n$ (= $4m$) bursts is evaluated, the "frozen" error matrices $\Lambda_k$ will be cancelled. Therefore the estimate (2) approaches

$$\mathbf{x} = \frac{2}{nN} \sum_{k=0}^{n-1} \mathbf{W}_k' \mathbf{y}_k \tag{3}$$

The covariance matrix associated with the estimate (3) is a diagonal matrix of order $2m$ with diagonal element [4]

$$u(\mathbf{x}) = \frac{2}{nN(nN-2m)} \cdot \left\{ \sum_{k=0}^{n-1} \mathbf{y}_k' \mathbf{y}_k - \frac{2}{nN} \left\| \sum_{k=0}^{n-1} \mathbf{W}_k' \mathbf{y}_k \right\| \right\} \tag{4}$$

where the symbol $\|\cdot\|$ denotes the Euclidean norm.

The DVM input stages and nonideal sampling introduce systematic errors that need to be corrected. The RMS magnitude of the $j$-th harmonic, corrected for all known systematic effects, is

$$V_j = \kappa k_{bw}(jf_0) k_{int}(jf_0) \cdot \left[\left((\mathbf{x})_j^2 + (\mathbf{x})_{m+j}^2\right)/2\right]^{1/2} \tag{5}$$

where κ is the correction of the DVM dc voltage mode error, $k_{bw}(jf_0)$ is the frequency response correction of the input stages, $k_{int}(jf_0)$ is the A/D converter frequency response correction, and the symbol $(\cdot)_j$ denotes the $j$-th element. The elements of $\mathbf{x}$ are uncorrelated. Assuming that the corrections are also uncorrelated, the uncertainty associated with the magnitude $d_j$ of the $j$-th harmonic ($j$ = 2, …, $m$) relative to the fundamental $V_1$ is

$$u(d_j) \approx \left\{ d_j^2 \left[ u^2(k_{bw}(jf_0)) + u^2(k_{bw}(f_0)) + u^2(k_{int}(jf_0)) + u^2(k_{int}(f_0)) \right] + (1 + d_j^2) \cdot u^2(\mathbf{x}) / 2V_1^2 \right\}^{1/2} \tag{6}$$

where the noise contribution is in general dominant.

## Performance Tests

A stable, high-resolution DVM controlled by the algorithm was used to measure the harmonic magnitudes of periodic arbitrary signals generated by a stable, digitally-synthesised, arbitrary signal generator (also used in [3]). The latter synthesises the signals in a staircase approximation. The values stored in memory are equally-spaced samples of these signals with 2048 discrete steps per period with 12-bit amplitude resolution. The shape of each signal is specified mathematically.

Several 60-Hz nonsinusoidal signals in the 10 V range were synthesised and separately applied to the DVM input. The Fourier coefficients relative to the fundamental were numerically evaluated for each waveform and compared with the algorithm output. Due to space limitation, only the results obtained for a half-wave rectified signal with $m = 42$ are described below.

The algorithm took about 3.75 min to evaluate the harmonic magnitudes. The reported THD was 43.5642%. The algorithm selected $n = 168$ bursts of $N = 167$ samples spaced by $t_{samp}$=0.0001996 s. The fundamental magnitude was measured with an uncertainty of 9.2 μV/V. The harmonic magnitudes relative to the fundamental were measured with an uncertainty of less than $6.6\times10^{-6}$. The computed and measured results are shown in Table I. No difference was detected at the estimates for $d_j$ when the frequency errors were within $\pm10^{-4}$. An uncertainty within these limits is easily attainable by signal null-crossing techniques.

## Conclusion

It was shown that an algorithm based on discrete Fourier transforms and Swerlein´s algorithm can be used to measure the harmonic magnitudes relative to the fundamental of arbitrary signals at low frequencies with an uncertainty of less than $1\times10^{-5}$. Periodic nonsinusoidal signals were synthesised by a commercial source and measured by the algorithm. The differences between computed and measured values suggest that stable, digitally-synthesised signal generators can be used as a calculable standard of harmonic distortion with an accuracy of less than $6\times10^{-5}$ relative to the fundamental.

## References

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**Table I.** Computed and measured values of the harmonic magnitudes of a 60-Hz half-wave rectified signal.

| $j$ | Magnitude (%) | | Error ($10^6$) | $j$ | Magnitude (%) | | Error ($10^6$) |
|---|---|---|---|---|---|---|---|
| | Comp. | Meas. | | | Comp. | Meas. | |
| 1 | 100.000 | 100.000 | - | 22 | 0.26405 | 0.26722 | 32 |
| 2 | 42.4838 | 42.4783 | –55 | 23 | 0.00402 | 0.00575 | 17 |
| 3 | 0.03444 | 0.02854 | –59 | 24 | 0.22185 | 0.22284 | 10 |
| 4 | 8.49489 | 8.49682 | 19 | 25 | 0.00371 | 0.00538 | 17 |
| 5 | 0.01914 | 0.01923 | 1 | 26 | 0.18902 | 0.18889 | –1 |
| 6 | 3.64069 | 3.63896 | –17 | 27 | 0.00343 | 0.00433 | 9 |
| 7 | 0.01340 | 0.01165 | –17 | 28 | 0.16298 | 0.16281 | –2 |
| 8 | 2.02270 | 2.02141 | –13 | 29 | 0.00320 | 0.00392 | 7 |
| 9 | 0.01034 | 0.00767 | –27 | 30 | 0.14199 | 0.14165 | –3 |
| 10 | 1.28726 | 1.28723 | –5 | 31 | 0.00300 | 0.00364 | 6 |
| 11 | 0.00843 | 0.00532 | –31 | 32 | 0.12481 | 0.12425 | –6 |
| 12 | 0.89126 | 0.89390 | 26 | 33 | 0.00282 | 0.00322 | 4 |
| 13 | 0.00712 | 0.00765 | 5 | 34 | 0.11057 | 0.10983 | –7 |
| 14 | 0.65366 | 0.65351 | –1 | 35 | 0.00266 | 0.00283 | 2 |
| 15 | 0.00616 | 0.00543 | –7 | 36 | 0.09865 | 0.09776 | –9 |
| 16 | 0.49992 | 0.49960 | –3 | 37 | 0.00252 | 0.00223 | –3 |
| 17 | 0.00544 | 0.00348 | –20 | 38 | 0.08856 | 0.08753 | –10 |
| 18 | 0.39472 | 0.39643 | 17 | 39 | 0.00240 | 0.00131 | –11 |
| 19 | 0.00487 | 0.00445 | –4 | 40 | 0.07994 | 0.08026 | 3 |
| 20 | 0.31959 | 0.32030 | 7 | 41 | 0.00228 | 0.00200 | –3 |
| 21 | 0.00440 | 0.00279 | –16 | 42 | 0.07253 | 0.07246 | –1 |